where N is given by $N \lesssim 1/\delta$ and δ is the angular width of the slot in radians.

Consider the slot admittant $Y(0)$ for an infinitesimally thin slot which is placed along the equator of a perfectly conducting sphere of radius a [1]

$$Y(0) = \sum_n^\infty Y_n(0) = \frac{j\pi}{\eta} \sum_n^\infty \frac{2n+1}{n(n+1)} \cdot \frac{[P_n^1(0)]^2}{\frac{n}{ka} - \frac{H_{n-\frac{1}{2}}^{(2)}(ka)}{H_{n+\frac{1}{2}}^{(2)}(ka)}}. \quad (1)$$

The series does not converge since finite contributions to the imaginary part of Y are added forever. The admittance for a slot of finite gap width $a\delta$, where $\delta \ll 1$ can be written as [2]

$$Y(\delta) = \sum_n^\infty Y_n(\delta) = \frac{j\pi}{\delta^2 \eta} \sum \frac{2n+1}{n(n+1)} \frac{\left[P_n\left(\cos\left[\frac{\pi}{2} + \frac{\delta}{2}\right]\right) - P_n\left(\cos\frac{\pi - \delta}{2}\right)\right]^2}{\frac{n}{ka} - \frac{H_{n-\frac{1}{2}}^{(2)}(ka)}{H_{n+\frac{1}{2}}^{(2)}(ka)}}. \quad (2)$$

It is seen that this expression is similar to $Y(0)$, except that the Legendre functions are those appropriate for a finite slot. We will now show that the first few terms of both series are approximately equal. This comes about since the P_n's show little variation for small δ in the range of $\pi/2 - \delta/2$ to $\pi/2 + \delta/2$ when n is not too large (recalling that n gives the number of zeros that the Legendre function goes through in the range of θ from 0 to π). Here we can expand

$$P_n \left\{\cos \frac{\pi \pm \delta}{2}\right\}$$

about $P_n(0)$ giving for

$$[P_n(+) - P_n(-)]^2 = [\delta P_n^1(0)]^2 \left(1 - \frac{n^2+n-1}{12}\delta^2 + \cdots\right). \quad (3)$$

Ignoring the δ^2 terms in the preceding Taylor expansion and then substituting it in (2) we obtain an approximate series for $Y(\delta)$ whose first few terms are those of the infinitesimal slot expression $Y(0)$. This approximation is valid for the first terms but becomes unusable for the higher terms. Let us denote by N the upper bound for the summation index n for which this approximation is valid, i.e.,

$$Y(\delta) = \sum_{n=1}^N Y_n(0) + \cdots. \quad (4)$$

The validity of the preceding Taylor expansion for P_n determines N. The zeros of the Legendre function are almost uniformly distributed in the range of θ from 0 to π. Thus the number of zeros within the range of δ should be approximately equal to $\delta n/\pi$. Now for the two-term Taylor expansion to be valid, the Legendre function must not vary much over the gap width. This can be stated in terms of the number of zeros which are included in the gap region as $\delta n/\pi < 1$. This condition ensures that the slot does not extend beyond the first maximum of $P_n(\cos\theta)$ on either side of $\theta = \pi/2$. Hence it follows that an upper bound for N can be taken as $N \approx 1/\delta$. This condition follows also immediately if in (3) the terms in the round bracket are not to differ from unity by more than ϵ, when ϵ is small, then we must have $n \lesssim \sqrt{12\epsilon}/\delta$. Hence for $n \leq N$, $Y_n(\delta) \simeq Y_n(0)$.

To evaluate the rest of the series (2) whose terms will show more and more variation over the gap width as n gets larger, we can make use of the asymptotic expression for P_n as n becomes large. Again retaining first-order terms the remainder of the series for the admittance of a finite slot can then be written as

$$\text{Remainder } Y(\delta) = \frac{j4ka}{\delta^2\eta} \sum_{n=N+2}^\infty \frac{2n+1}{n^3(n+1)}(1 - \cos n\delta) = \sum Y_\infty(\delta) \quad (5)$$

where the Hankel function terms in the denominator of Y are approximated for large n as n/ka. The terms in the remainder series converge as $1/n^3$, whereas the remaining terms in the series for the infinitesimal slot diverge. An estimate of the remainder (5) can be obtained by noting that

$$\sum_{n=N+2}^\infty Y_n(\delta) < M \sum_{n=N+2}^\infty 1/n^3 \quad (6)$$

where M is a constant. The series of $1/n^3$ terms can be summed by converting to a contour integral with poles along the real axis, which can then be deformed to a path parallel to the imaginary axis. By changing variables the resulting integral can be estimated yielding

$$\sum_{n=N+2}^\infty 1/n^3 < C \frac{1}{(N+2)^2} \quad (7)$$

where C is another constant. Thus the slot admittance can be written as

$$Y(\delta) = \sum_{n=1}^N Y_n(0) + O(1/N^2). \quad 8)$$

In case the exact value of the remainder series is desired,

$$\sum_{n=1}^\infty Y_\infty(\delta)$$

can be evaluated in closed form [3]. The remainder (5) can then be obtained by subtracting the sum of the first N terms, and then can be used to approximate the exact remainder to (2).

Thus it can be concluded that the admittance of a finite slot can be reasonably approximated by the generally divergent expression for the admittance of an infinitesimally narrow slot by summing the series to the first N terms, where N is given by $N \approx 1/\delta$, and δ is the angular width of the slot in radians. The error created in leaving off the remaining terms is then of the order $O(1/N^2)$.

M. A. Plonus
Dept. of Elec. Engrg.
Northwestern University
Evanston, Ill.

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Propagation of the Quasi-TEM Mode in Ferrite-Filled Coaxial Line

Brodwin and Miller,[1] in discussing the propagation of the quasi-TEM mode in ferrite-filled coaxial line, were apparently unaware of an earlier comprehensive treatment of the subject.[2,3] For purpose of comparison, the notations in this correspondence correspond to those used by Brodwin and Miller.

In the earlier treatment, the Suhl and Walker approximation for parallel plane geometry was extended to coaxial geometry by requiring that the conditions

$$|S_1| R_2 \ll 1, \quad |S_2| R_2 \ll 1, \quad |S_1| R_1 \ll 1, \quad |S_2| R_1 \ll 1 \quad (1)$$

be satisfied where R_2 and R_1 are the outer and inner radii, respectively, of the coaxial line.[4,5] If conditions (1) are substituted directly into the exact determinantal equation for the quasi-TEM mode in coaxial geometry,[4] the determinantal equation reduces to[5]

$$\gamma_0^2 = -\omega^2\epsilon \left[\frac{\mu^2 - k^2}{\mu}\right] \quad (2)$$

for nontrivial values of S_1 and S_2.

For parallel plane geometry, (2) is known as the Suhl and Walker approximation to the propagation constant of the quasi-TEM mode. It can be shown by direct substitution into the exact determintal equation that (2) is valid in parallel plane

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[1] M. E. Brodwin and D. A. Miller, "Propagation of the quasi-TEM mode in ferrite-filled coaxial line," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 496–503, September 1964.

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[4] R. Teragawa et al., *op. cit.*,[2] Interim Development Rept. 1, June–August 1956, pp. 6–12.

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TABLE I

H_n	R_2/R_1	S_1R_1	S_2R_1	S_1R_2	S_2R_2	Percent Deviation*
0.75	2.25	1.06	j 0.65	2.40	j 1.46	3.7 percent
1.2	2.25	0.383	j 0.311	0.86	j 0.70	0.4 percent
1.2	6.0	0.383	j 0.311	2.39	j 1.87	3.8 percent
2.0	2.25	0.196	j 0.177	0.44	j 0.40	<0.1 percent
2.0	7.0	0.196	j 0.177	1.37	j 1.24	0.7 percent
3.0	2.25	0.126	j 0.119	0.28	j 0.27	<0.03 percent
3.0	7.0	0.126	j 0.119	0.88	j 0.83	not calculated

* Percent deviation of Suhl and Walker Approximation from numerical solution.

geometry for either set of the following conditions:

$$|S_1x_2| \ll 1, \quad |S_2x_2| \ll 1, \quad |S_1x_1| \ll 1, \quad |S_1x_1| \ll 1 \quad (3)$$

or

$$|S_1| \, |x_2 - x_1| \ll 1, \quad |S_2| \, |x_2 - x_1| \ll 1 \quad (4)$$

where x_2 and x_1 are the positions of the higher and lower parallel plates, respectively. Since the restriction $x_1 = 0$ does not alter the parallel plane geometry, conditions (3) and (4) may be considered as equivalent conditions.

For coaxial geometry, conditions (1) automatically imply

$$|S_1| (R_2 - R_1) \ll 1, \quad |S_2| (R_2 - R_1) \ll 1 \quad (5)$$

but (5) does not necessarily imply (1) since $R_1 \neq 0$. If conditions (5) are substituted directly into the determinantal equation with the additional constraint that conditions (1) are violated, the determinantal equation becomes identically zero regardless of whether the propagation constant is given by (2) or not. The fact that conditions (5) do not uniquely define the propagation of the quasi-TEM mode is not too surprising, since "unrolled" coaxial line is not perfectly analogous to parallel plane geometry. In parallel plane geometry, higher order modes are cut off for sufficiently small spacing between planes whereas in large radius coaxial line higher order modes can propagate despite the spacing between conductors.[6]

Conditions (1) are both necessary and sufficient conditions for propagation of the quasi-TEM mode in coaxial geometry so that all other modes are cut off. Under these conditions, the propagation constant is given approximately by (2). The solution for the quasi-TEM mode in coaxial geometry was first reported in 1957[5] and subsequently in 1959.[3]

M. M. WEINER
Advanced Research Dept.
Edgerton, Germeshausen, and Grier, Inc.
Boston, Mass.

[6] S. Ramo and J. R. Whinnery, *Fields and Waves in Modern Radio*, 2nd ed. New York: Wiley, p. 365.

Authors' comment[7]

We wish to thank Mr. Weiner for bringing his work to our attention. While it is true that the determinantal equation does reduce to zero for small arguments if the Suhl and Walker approximation holds, we do not believe that the small argument restriction is necessary. As reported in our paper, the Suhl and Walker approximation was found to deviate from numerical solutions of the determinantal equation by only a few percent over a wide range of magnetic fields and outer radii for a given material and inner radius.

Table I summarizes calculations which demonstrate that, even for large arguments, the Suhl and Walker approximation is quite useful.

These calculations were carried out for the following parameters: frequency 1.5 Gc/s, $4\pi M_s = 680$ q., $\epsilon_r = 11.5$, $R_1 = 0.125$ inch. The magnetic field is normalized to the resonant field, 536 Oe, and the ratio $R_2/R_1 = 2.25$ corresponds to 50-ohm, air-filled coaxial line.

The first line of Table I presents an example in which the arguments are large, but the deviation is small enough for engineering purposes.

M. E. BRODWIN
Northwestern University
Evanston, Ill.
D. MILLER
Illinois Institute of Technology
Chicago, Ill.

[7] Manuscript received February 1, 1965; revised September 1, 1965.